Harold ERBIN

Talk : Initial value problem and causality in string-inspired non-local field theory

Collaborators : Atakan Hilmi Fırat, Barton Zwiebach
arxiv: 2111.03672

Plan

1. Motivations
2. Model
3. Field redefinition of the time-only theory
4. Field redefinition of the general theory
5. Rolling tachyon analysis
6. (Causality from dispersion relations)

In a nutshell

- string theory is non-local
- field redefinitions allow to make the action local in time, with only first-order derivatives (but non-local in space)
- tame oscillations of rolling tachyon solution
- massive scalar / tachyon in constant background shows no superluminal propagation

1. Motivations

- string theory interactions are non-local since they contain
  $$e^{-ck^2} = e^{c\partial^2}$$

- covariant formulation: time and space non-locality

- ensures UV finiteness (through generalized Wick rotation)
  [Buis-Yen, 1604.01783]

- time non-locality is problematic
  - definition of Hamiltonian?
  - initial value formulation?
  - causality violation?

- important to understand physical properties
  - differences with local QFT?
  - relation to black hole information paradox?

What do we mean by causality?
- microcausality = fields commute at spacelike separations
- Bogoliubov causality condition
- equation of motion has well-defined initial value problem
- retarded Green function has support in past light-cone only
- absence of superluminal propagation
  (from dispersion relations and PDE characteristics)
- chronology condition: absence of closed timelike curves
- absence of Shapiro time advance
- (macrocausality) (= asymptotic causality = amplitude fall-off + stable asymptotic states)

$[\varphi(x), \varphi(y)] = 0$ for $(x-y)^2 \geq 0$

$G_R(x,y) \neq 0$ for $\begin{cases} (x-y)^2 \leq 0 \\ y^0 \geq x^0 \end{cases}$

Problems with higher-order time derivatives:

- Ostrogradski instability
  → only for finite # of derivatives
  procedure not working for infinite #
  (related to lack of Hamiltonian)

- no Hamiltonian (how to define conjugate momentum?)
  Ostrogradski construction fails
  possible solution: smeared Hamiltonian over time slice (~ non-locality scale)
  [Barnaby, 1005.2945; Tomboulis, 1507.00981]

Hayashi

- differential equation: each order in time derivative = 2 initial conditions
  infinite-order $\overset{?}{\Longrightarrow}$ infinitely many conditions = loss of predictivity
  answer is no, only singularities matter [Barnaby-Kamran, 0709.3968]

Some observations for SFT: many studies but no clear conclusion

- light-cone SFT local in lightcone time x^+ (even 1st order)

- same for Witten open SFT in lightcone basis [Erler-Gross, hep-th/0406199]
  proof that light-cone SFT = gauge fixing of Witten SFT
  [Erler-Matsunaga, 2012.09521]

kinetic term = local

- perturbative localization: perturbation theory reduces phase space
  (filters out non-perturbative solutions, changes canonical structure, can write perturbative local action) [Eliezer-Woodard '89]
  → lightcone actions

- in local QFT, spacelike commutativity ⇒ primitive analyticity
  SFT: analyticity still holds
  [de Lacroix-HE-Sen, 1810.07197]
  [Bhattacharya-Mahanta, 2009.03375, 2110.13125]

Questions:

- understand better the status of causality and non-locality in SFT
- find formulations making clear that the initial value problem is well-posed
- revisit rolling tachyon

2. Model

Open string tachyon $\varphi(x)$:

$$p^2 = -m^2 = \frac{1}{\alpha'}$$

$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \dots)$

$\partial^2 = -\partial_t^2 + \vec{\nabla}^2$

Witten SFT truncated to tachyon:

$$L = \frac{1}{2}\varphi(\alpha'\partial^2 + 1)\varphi + \frac{g}{3}\left(e^{\xi^2(\alpha'\partial^2+1)}\varphi\right)^3$$

Non-locality parameter:

$$\xi^2 := \ln\frac{3\sqrt{3}}{4} \simeq 0.26$$

Non-locality scale:

$$\ell^2 := \frac{\xi^2}{m^2} = \alpha'\xi^2 \quad \text{(tachyon)}$$

p-adic string: $\xi_p^2 = \frac{1}{4}\ln p$ $\quad (\xi_2^2 \simeq 0.17)$

Rescalings: $\alpha'\partial \to \partial$, $e^{\xi^2} g \to g$, $\varphi \to \varphi/g$, $L \to g^2 L$

New form

$$L = -\frac{1}{2}\varphi(-\partial^2 + m^2)\varphi + \frac{1}{3}\left(e^{\xi^2\partial^2}\varphi\right)$$

treat $m^2 \in \mathbb{R}$ and $\xi^2 > 0$ as parameters for analytic control

note: only sign m^2 matters $(\varphi \to \varphi/m,\ \partial \to m\partial,\ \xi \to \xi/m)$

Remarks:

- ξ^2 expansion = derivative exp. = α' exp.
  $\to$ effective field theory

- higher-derivative = 2+ derivatives acting on one field
- higher derivative term = term in action which contains higher-deriv. which cannot be removed by integrating by part

3. Time-only theory

$$L = -\frac{1}{2}\varphi(\partial_t^2 + m^2)\varphi + \frac{1}{3}\left(e^{-\xi^2\partial_t^2}\varphi\right)^3$$

Claims:

1. Higher-derivatives can be removed by field redefinitions.
2. All terms of the form $\varphi^k \dot{\varphi}^n$ with $k \geq 0$, $n \geq 0$ can be removed.

$\Rightarrow$ The redefined L has a canonical kinetic term and only a potential.

Perform a field redefinition:

$$\hat{L}[\varphi] = L[\varphi + \delta\varphi]$$

$$= L[\varphi] - \delta\varphi(\partial_t^2 + m^2)\varphi + \delta\varphi\, e^{-\xi^2\partial_t^2}\left(e^{-\xi^2\partial_t^2}\varphi\right)^2$$

$$-\frac{1}{2}\delta\varphi(\partial_t^2 + m^2)\delta\varphi + \left(e^{-\xi^2\partial_t^2}\delta\varphi\right)^2 e^{-\xi^2\partial_t^2}\varphi + \frac{1}{3}\left(e^{-\xi^2\partial_t^2}\delta\varphi\right)^3$$

Idea: since 2nd term contains $\partial_t^2\varphi$, can remove derivatives

Expand the exponential, field redefinition and Lagrangian:

$$e^{-\xi^2\partial_t^2}\varphi = \sum_{n\geq 0}\frac{(-1)^n}{n!}\xi^{2n}\partial_t^{2n}\varphi \qquad \delta\varphi = \sum_{n\geq 1}\xi^{2n}\delta_{2n}\varphi$$

$$L = \sum_{n\geq 0}\xi^{2n}L_{2n} \qquad \tilde{L} = \sum_{n\geq 0}\xi^{2n}\tilde{L}_{2n}$$

$$L_0 = -\frac{1}{2}\varphi\partial_t^2\varphi + V(\varphi), \qquad V(\varphi) = -\frac{m^2}{2}\varphi^2 + \frac{1}{3}\varphi^3$$

$$L_2 = -\varphi^2\partial_t^2\varphi$$

Consider the lowest order:

$$L_0[\varphi+\delta\varphi] = L_0[\varphi] - \xi^2 \delta_2\varphi \left(\partial_t^2\varphi + V'(\varphi)\right) + O(\xi^4)$$

$$\Rightarrow \tilde{L}_2 = -\varphi^2\partial_t^2\varphi - \delta_2\varphi\left(\partial_t^2\varphi + V'(\varphi)\right)$$

$$= -\partial_t^2\varphi\left[\delta_2\varphi + \varphi^2\right] - \delta_2\varphi\, V'(\varphi)$$

Can remove higher-order derivative:

$$\boxed{\begin{aligned} \delta_2\varphi &= -\varphi^2 \\ \tilde{L}_2 &= -\varphi^2 V'(\varphi) = \frac{m^2}{2}\varphi^3 - \frac{1}{3}\varphi^4 \end{aligned}}$$

note: $\varphi^2\partial_t^2 \simeq -2\varphi\dot{\varphi}^2$
so not higher-order deriv.
but same idea

The field redefinition contributes to $O(\xi^4)$ terms, which must also be removed.

We work recursively: at each order, we have derivative terms from:
1. expansion of the exponentials
2. non-linear contributions from field redef. for lower orders

Assume that we removed all derivatives up to $O(\xi^{2k-2})$:

$$\tilde{L} = L_0 - \xi^2\tilde{V}_2 - \cdots - \xi^{2k-2}\tilde{V}_{2k-2} + \xi^{2k} L'_{2k} + O(\xi^{2k+2})$$

where $\tilde{L}_{2n} = -\tilde{V}_{2n}$ for $n<k$ are the redefined terms and polynomial in φ

$$L'_{2k}[\varphi] = L_{2k}[\varphi] + L\left[\varphi + \sum_{n=0}^{k-1}\delta_{2k}\varphi\right]\Big|_{\xi^{2k}}$$

Performing a field redef:

$$\varphi \to \varphi + \xi^{2k}\delta_{2k}\varphi$$

gives:

$$\tilde{L} = L_0 - \xi^2\tilde{V}_2 - \cdots - \xi^{2k-2}\tilde{V}_{2k-2} + \xi^{2k}\left[L'_{2k} - \delta_{2k}\varphi\left(\partial_t^2\varphi + V'(\varphi)\right)\right]$$

$$+ O(\xi^{2k+2})$$

Only terms linear in $\delta_{2k}\varphi$ appear.
Since we have $\delta_{2k}\varphi\, \partial_t^2\varphi$, we can remove any term of the form

$$T = X(\varphi)\,\partial_t^2\varphi \longrightarrow \partial_t^2\varphi\left(X(\varphi) + \delta_{2k}\varphi\right) + V'(\varphi)\,\delta_{2k}\varphi$$

$$= -V'(\varphi)\,X(\varphi)$$

for $\delta_{2k}\varphi = -X(\varphi)$

The total $\delta_{2k}\varphi$ is the sum of all such redefinitions.

3-step recursive algorithm:

1. Remove all terms containing $\partial_t^2\varphi$.

2. General term has the form:

$$T = (\partial_t^{k_1}\varphi)\cdots(\partial_t^{k_\ell}\varphi)\,(\partial_t\varphi)^r\,\varphi^s$$

$k_\ell \geq \cdots \geq k_1 \geq 3$, $r, s \geq 0$
number of higher deriv $:=$ degree
We need to show that we can reduce the degree ℓ of any term.

a) IPP:

$$T \simeq -(\partial_t^{k_1-1}\varphi)\,\partial_t\left[(\partial^{k_2}\varphi)\cdots(\partial^{k_\ell}\varphi)\right](\partial_t\varphi)^r\varphi^s$$

$$+ (\partial_t^{k_1-1}\varphi)(\partial^{k_2}\varphi)\cdots(\partial^{k_\ell}\varphi)(\partial_t\varphi)^{r-1}\varphi^{s-1}\left[\varphi\,\partial_t^2\varphi + (\partial_t\varphi)^2\right]$$

↳ remove

The lowest-order has been decreased.

b) Iterate: at some point, we will get $\partial_t^2\varphi$, and the term can be redefined away. This decreases the degree by 1.

c) Repeat the procedure for the other higher-deriv. terms in T.

3. After removing all the higher-derivatives terms, a general term has the form:

$$T' = (\partial_t \varphi)^p \varphi^q \simeq -q (\partial_t \varphi)^p \varphi^{q-1} - (p-1)(\partial_t \varphi)^{p-2} \varphi^{q+1} \partial_t^2 \varphi$$

$$\simeq -\frac{p-1}{q+1} (\partial_t \varphi)^{p-2} \varphi^{q+1} \partial_t^2 \varphi$$

$$\longrightarrow \frac{p-1}{q+1} (\partial_t \varphi)^{p-2} \varphi^{q+1} V'(\varphi)$$

$\longrightarrow$ remove recursively all first-order deriv.

Result:

$$\tilde{L} = -\frac{1}{2} \varphi \partial_t^2 \varphi - \tilde{V}(\varphi; \xi^2)$$

$$\tilde{V} = \frac{m^2}{2} \varphi^2 - \left[\frac{1}{3}\varphi^3 + m^2 \xi^2 + \frac{3}{2} m^4 \xi^4 + \dots\right] \varphi^3$$
$$+ \left[\xi^2 + \frac{19}{3} m^2 \xi^4 + \dots\right] \varphi^4 + \left[-\frac{16}{3} \xi^4 + \dots\right] \varphi^5 + O(\xi^6)$$

Note: field redefinition must preserve value of potential at critical points

$$V'(\varphi_*) = 0 \implies \varphi_* = m^2, \quad V(\varphi_*) = \frac{m^2}{6}$$

$$\tilde{V}'(\tilde{\varphi}_*) = 0 \implies \tilde{V}(\varphi_*) = \frac{m^2}{6}$$

Reason: $\delta \varphi = f(\varphi) + g(\varphi, \dot{\varphi}, \dots)$
$\tilde{V}(\varphi) = V(\varphi + f(\varphi))$

However, $\tilde{V}(\varphi, \xi^2)$ is ambiguous.

Quasi-symmetries

Two observations:
- for a local theory, changing the potential changes the kinetic term
- we have ignored total derivatives

Consider the redef:

$$\delta\varphi = \dot\varphi^2 + V(\varphi)$$

$$\tilde L = L - \delta\varphi\,(\ddot\varphi + V'(\varphi)) + O(\delta\varphi)^2$$

$$= L - (\dot\varphi^2 + V(\varphi))(\ddot\varphi + V'(\varphi)) + O(\delta\varphi^2)$$

$$= L \underbrace{- \partial_t(\dot\varphi^3) - \partial_t(\dot\varphi V)}_{\text{total derivatives}} - VV' + O(\delta\varphi^2)$$

At the linearized order, $\delta\varphi$ only changes the potential

$$\delta V = VV' = \frac{m^4}{2}\varphi^3 - \frac{5m^2}{6}\varphi^4 + \frac{\varphi^5}{3}$$

Alternative point of view: add total derivative and apply algo

$$L \longrightarrow L + \partial_t(\dot\varphi^3) = L + 3\dot\varphi^2\ddot\varphi$$

Non-linear terms in $\delta\varphi$ contributes to higher-orders in ξ^2 only, and can be removed.

$\longrightarrow$ effective symmetry of the action

Most general quasi-symmetry:

$$\delta\varphi = \sum_{n\geq 1}\left(\dot\varphi^{2n} + (2n-1)\dot\varphi^{2n-2}V + \dots + \frac{(2n-1)!!}{n!}V^n\right)$$

Use quasi-symmetries to write coeff. of odd-powers φ^n as finite polynomial in ξ^2:

$$\hat{V}(\varphi^2;\xi^2) = \frac{m^2}{2}\varphi^2 - \left[\frac{1}{3} + m^2\xi^2\right]\varphi^3$$
$$+ \left[\xi^2 + \frac{23}{6}m^2\xi^4 + \frac{112}{9}m^4\xi^6 + \frac{400}{9}m^6\xi^8 + \dots\right]\varphi^4$$
$$- \left[\frac{13}{3}\xi^4 + \frac{370}{9}m^2\xi^6 + \frac{2356}{9}m^4\xi^8\right]\varphi^5$$
$$+ \dots + O(\xi^{10})$$

Notes:

- computed up to ξ^{22}
- same critical depth
- seems to have finite radius of convergence for $\xi^2 < 0.5$

4. Full covariant theory

$$L = -\frac{1}{2}\varphi(-\partial^2+m^2)\varphi + \frac{1}{3}\left(e^{\xi^2\partial^2}\varphi\right)^3$$

Claims:

- can be made local up to $O(\xi^6)$
- obstruction at $O(\xi^8)$
  $\longrightarrow$ can be removed if give up covariance

Note: cannot remove first-order term except $\varphi^n(\partial\varphi)^2$

We have:

$$\tilde{L}[\varphi] = L[\varphi+\delta\varphi] = L[\varphi] + \delta\varphi\left(\partial^2\varphi - V'(\varphi)\right) + O(\delta\varphi^2)$$

$\longrightarrow$ apply the same algorithm as before, but also integrate $\vec{\nabla}$ by part to extract $X\partial^2\varphi$

Result:

$$\tilde{L} = -\tilde{K}(\varphi,\partial\varphi;\xi^2) - \tilde{V}(\varphi;\xi^2) - \frac{4}{3}(\partial\varphi)^2\partial^2(\partial\varphi)^2$$

$$\tilde{K} = -\frac{1}{2}(\partial\varphi)^2 + \left[\frac{8}{3}\xi^6 + \frac{52}{3}m^2\xi^8\right](\partial\varphi)^4 - 96\xi^8\varphi(\partial\varphi)^4 + O(\xi^{10})$$

$$\tilde{V} = \frac{m^2}{2}\varphi^2 - \frac{1}{3}e^{3m^2\xi^2} \quad \text{(exact!)}$$
$$+\left[\xi^2 + \frac{19}{3}m^2\xi^4 + \frac{178}{9}m^4\xi^6 + \frac{223}{6}m^6\xi^8\right]\varphi^4$$
$$-\left[\frac{16}{3}\xi^4 + \frac{472}{9}m^2\xi^6 + \frac{483}{2}m^4\xi^8\right]\varphi^5$$
$$+\left[\frac{112}{3}\xi^6 + 485\,m^2\xi^8\right]\varphi^6 - \frac{2695}{9}\xi^8\varphi^7 + O(\xi^{10})$$

Note: checked matching of 4-pt amplitude to $O(\xi^6)$

Before removing $\varphi^n(\partial\varphi)^2$, $\tilde{L}$ is of k-inflation type up to $O(\xi^6)$.
$\longrightarrow$ can use result on causality from there
[Garriga-Mukhanov, hep-th/9904176;
Babichev-Mukhanov-Vikman, 0708.0561]

The term $(\partial\varphi)^2\partial^2(\partial\varphi)^2$ cannot be written as $X\,\partial^2\varphi$ by IPP.
Solution: breaks Lorentz covariance.

1. Split time and space, redefine only higher-time derivatives.

$$\delta\varphi\,\partial^2\varphi = \delta\varphi\left(-\ddot{\varphi} + (\vec{\nabla}\varphi)^2\right)$$

Result: talk: edit previous

$$\tilde{L} = -\tilde{K}(\varphi, \partial\varphi; \xi^2) - \tilde{V}(\varphi; \xi^2)$$

$$\tilde{K} = -\frac{1}{2}(\partial\varphi)^2 + \left[\frac{8}{3}\xi^6 + 20m^2\xi^8\right](\partial\varphi)^4 - \frac{304}{3}\xi^8\,\varphi\,(\partial\varphi)^4 + O(\xi^{10})$$
$$+ \frac{8}{3}\xi^8(\partial\varphi)^2\left[(\vec{\nabla}^2\varphi)^2 + (\nabla_i\nabla_j\varphi)^2 - 2(\vec{\nabla}\dot{\varphi})^2 + 2(\varphi^2 - m^2\varphi)\vec{\nabla}^2\varphi\right]$$

$$\tilde{V} = \frac{m^2}{2}\varphi^2 - \frac{1}{3}e^{3m^2\xi^2}$$
$$+ \left[\xi^2 + \frac{19}{3}m^2\xi^4 + \frac{178}{9}m^4\xi^6 + \frac{653}{18}m^6\xi^8\right]\varphi^4$$
$$- \left[\frac{16}{3}\xi^4 + \frac{472}{9}m^2\xi^6 + \frac{4307}{18}m^4\xi^8\right]\varphi^5$$
$$+ \left[\frac{112}{3}\xi^6 + \frac{7247}{15}m^2\xi^8\right]\varphi^6 - \frac{13451}{45}\xi^8\varphi^7 + O(\xi^{10})$$

Notes:
- can be carried to all orders
- only $\dot{\varphi}^n$ left, but arbitrary higher-spatial derivatives
- can write Hamiltonian (without Ostrogradski)
- well-defined initial value problem

2. Light-cone coordinates, redefine only light-cone time derivatives

Coordinates: $x^+ = \tau$, x^-, $\vec{x}_T$

$$L_0 = \frac{1}{2}\varphi\left(-\partial_-\partial_\tau + \vec{\nabla}_T^2 - m^2\right)\varphi + \frac{1}{3}\varphi^3$$

Field redefinition:

$$L_0[\varphi+\delta\varphi] = L_0[\varphi] + \delta\varphi\left[-2\partial_-\partial_\tau\varphi + \vec{\nabla}_T^2\varphi - m^2\varphi + \varphi^2\right] + O(\delta\varphi^2)$$

In light-cone, ∂_- is invertible, hence:

$$X\partial_\tau\varphi = X\frac{1}{\partial_-}(\partial_-\partial_\tau\varphi) \simeq -\partial_-\partial_\tau\varphi\frac{1}{\partial_-}X$$

$$\longrightarrow -\left(\vec{\nabla}_T^2\varphi - m^2\varphi + \varphi^2\right)\frac{1}{2\partial_-}X \simeq X\frac{1}{2\partial_-}\left(\vec{\nabla}_T^2\varphi - m^2\varphi + \varphi^2\right)$$

Using the same algorithm, we see that we can remove all $\partial_\tau^n\varphi$ and recover an action linear in $\partial_\tau\varphi$ as is the case for usual light-cone action.

5. Rolling tachyon

Non-local eq. of motion: take $m^2 = -1$

$$(\partial_t^2 - 1)\varphi = e^{-\xi^2\partial_t^2}\left(e^{-\xi^2\partial_t^2}\varphi\right)^2$$

Series solution:

$$\varphi(t) = \sum_{n\geq 1} b_n e^{nt}$$

$$b_n = \frac{1}{n^2-1}\sum_{p=1}^{n-1} b_p b_{n-p} e^{-2\xi^2(n^2-np+p^2)}$$

$$b_1 = -1, \quad b_2 = \frac{1}{3}e^{-6\xi^2}, \quad b_3 = -\frac{1}{12}e^{-20\xi^2}$$

We have:

$$b_n \simeq (-1)^n 16 n^2 e^{-\beta n - 3\xi^2 n^2} \qquad \beta \simeq 2$$

$\longrightarrow$ series converge at all time

But for finite n, solution displays growing oscillations after overshooting the turning point.

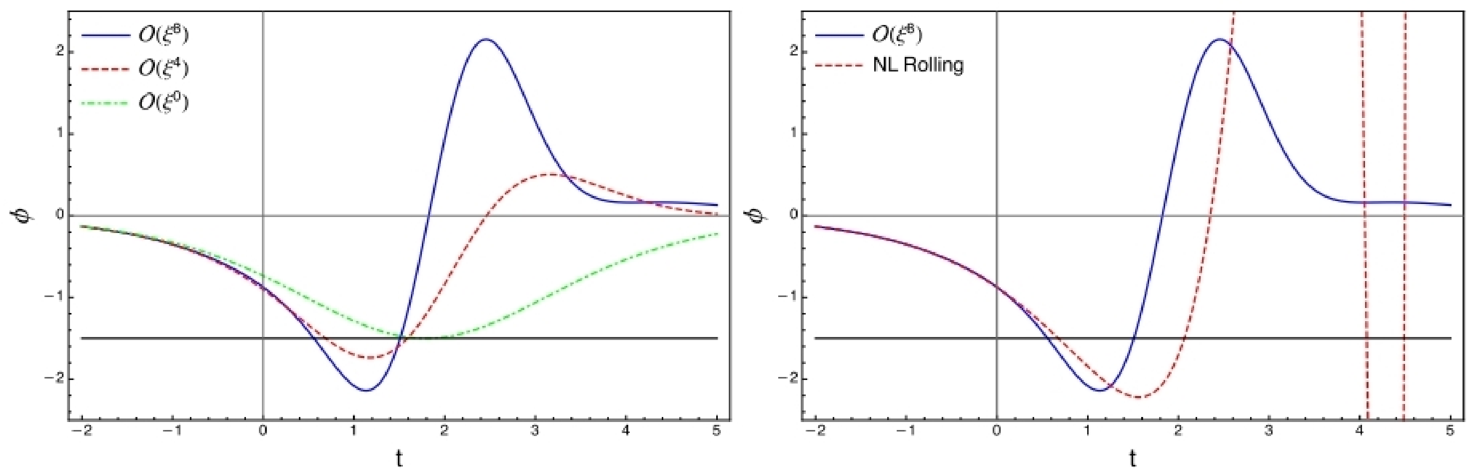
For $\xi^2 = 0$, the series converges only until the turning point.
But can resum / solve analytically:

$$\varphi_0(t) = -\frac{36 e^t}{(6+e^t)^2}$$

Can be used for perturbative solution in ξ^2:

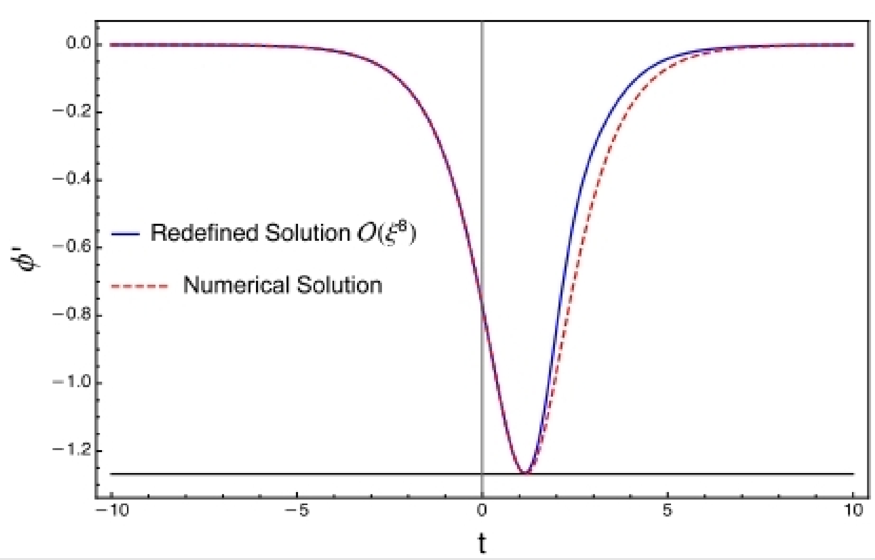
$$\varphi(t) = \varphi_0(t) + \sum_{n\geq 1}\xi^{2n}\varphi_{2n}(t)$$

$$\varphi_2(t) = \frac{432\, e^{2t}(e^t-6)}{(6+e^t)^4}$$



$\xi = 0.4$, $n \lesssim 14$ (red solution, right fig.)
turning point: $\varphi = -3/2$ (overshoot)

Map the solution (pert. in ξ^2) under the field redef.
$\longrightarrow$ compare with numerical solution for potential $\tilde{V}(\varphi;\xi^2)$



The field redefinition is doing the right thing and map to a nice solution in the new variable.

Interpretation?
- unstable vacuum (D-brane) $\longrightarrow$ tachyon vacuum (no D-brane)

- seems inconsistent with original solution (not symmetric) but numerical precision not sufficient
- not expected SFT behavior (final state = tachyon matter) (but missing massive states, etc.)

zero-pressure state

6. Conclusion

Results:

- field redefinitions allow to get action local in time (and at most first-order)
- implies existence of initial value problem and Hamiltonian
- suggest causality of SFT

Future directions:

- find closed-form expressions for $\tilde{V}(\varphi, \xi^2)$
- deeper analysis of causality (Bogoliubov condition, position-space representation...)